

A Mathematician's Viewpoint to Bell's theorem: In Memory of Walter Philipp

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Abstract

In this paper dedicated to the memory of Walter Philipp, we formalize the rules of classical \rightarrow quantum correspondence and perform a rigorous mathematical analysis of the assumptions in Bell's NO-GO arguments.

I met Walter Philipp on many occasions — mostly during the Växjö conferences and during my visits to the University of Illinois in Urbana-Champaign — and I always enjoyed social and scientific contacts with him. It was impressive that he always behaved as if he had just moved recently from Vienna (which in fact he had left as early as the 60th to move to Illinois): he was a man of great European cultural level, with a deep sense of humor, and he exhibited them both through uncountable stories about writers, poets, artists and scientists from Vienna.

Our common scientific interest was the mathematical formalization of Bell's arguments [1] which are widely known as Bell's NO-GO theorem. Contacts with Walter were very attractive for me, because we both had the same background: a specialization in probability theory. I was really happy to find in Walter a fellow mathematician with whom discussions on Bell's theorem could be made in the language of mathematically rigorous statements. Walter and I shared the common viewpoint — one that I have been trying to

advocate to physicists since the first conference in Vaxjo, on "Foundations of Probability and Physics" [2, 3, 4] — that without a rigorous mathematical formalization of the probabilistic content of Bell's arguments one cannot forcefully derive the fundamental dilemma that we are often being offered: that is, *either nonlocality or the death of reality* [5, 6, 7, 8, 9].

Our main point was that any mathematical theorem (when formulated in rigorous mathematical terms) is based on a list of assumptions. If such a precise list is not provided, then one cannot call it a mathematical theorem, and should not make any definitive conclusions. It was pointed out on many occasions, both by Walter Philipp and his collaborator Karl Hess, see e.g. [10]–[13], as well as by Luigi Accardi, see e.g. [14, 15], and myself [16]–[19], that without a presentation of a precise probabilistic model for Bell's framework, one cannot proceed in a rigorous way.

If one uses the Kolmogorov measure-theoretic model then one should be aware that there is no reason, even in classical physics, to assume that statistical data that were obtained in different experiments should be described by *a single Kolmogorov probability space*, see e.g. [16] for details. Walter Philipp strongly supported this kind of counterarguments by finding a purely mathematical investigations in probability which were devoted to a similar problem, but without any relation to quantum physics. In particular, Walter found a theorem (proved by a Soviet mathematician Vorob'ev[20]) describing the conditions which are necessary and sufficient for the realization of a few random variables on a single Kolmogorov space.

We remark that *Vorob'ev's theorem* was proved before the inequality which is nowadays known as Bell's inequality appeared in quantum physics. Vorob'ev applied his results to game theory. He noticed: "In application to the theory of games the foregoing means that a combination of players into groups can be found, the mixed coordinated actions of these groups cannot determine any mixed manner of action for all the players of the game which is consistent with the actions of these groups. The types of questions, developed in line with the requirements of the theory of coalition games, can find future applications also in the theory of information and the theory of random algorithms," [20], p. 147-148.

Unfortunately, studies of N. N. Vorob'ev were not supported by mathematical probabilistic establishment. In modern probability theory people traditionally operate in a single Kolmogorov probability space. Typically a mathematical paper in probability is started with the sentence: "Let $\mathcal{P} = (\Omega, \mathcal{F}, \mathbf{P})$ be a Kolmogorov probability space." Then everything should

happen in this chosen once and for ever probability space. When I started to develop so called contextual probability theory, i.e., a theory of probability in that any context (a complex of physical or biological, or social conditions) determines its own probability space [21], [22], the former students of Kolmogorov, Albert Shiryaev and Alexander Bulinskii, paid my attention to the fact that Kolmogorov by himself always underlined the role of a context in determining a probability space of an experiment [23], [24], see also Gnedenko [25] and Shiryaev [26].

Walter Philipp and his collaborator Karl Hess demonstrated [10]–[13] that for some extended models containing time as one of hidden variables the using of a single Kolmogorov probability space was not justified.

However, in physical literature it was often pointed out that in Bell's framework one need not use the Kolmogorov measure-theoretic model at all, because *it is possible to operate just with frequencies*. As was remarked first by De Baere [27] and then by Khrennikov [16], the frequency derivation of Bell's inequality is also based on mixing of statistical data from different experiments. Such a procedure was not totally justified. If we proceed in a rigorous mathematical way by using the von Mises frequency approach (which was recently presented on the mathematical level in [16]), then we shall immediately see that it is impossible to obtain Bell's inequality without additional assumptions. It is amazing that very soon Walter Philipp came independently to the same conclusion. It seems that everybody who was educated in probability theory should see such a problem.

We remark that there can be obtained generalized Bell-type inequalities which are not violated by the experimental statistical data taken from different experiments [16]–[18].

In this paper I would like to continue the "mathematical line" in analysing Bell's arguments. I would like to formalize the rules of classical→quantum correspondence. And we shall see that it is not so simple task. The first step in this direction was done by von Neumann when he formulated the first NO-GO statement for existence a prequantum classical statistical model [28]. J. Bell continued this activity by starting with heavy critical arguments against von Neumann's formalization. We proceed in the same way. Our analysis showed that Bell's formalization was far from to be complete.

I think such a mathematical analysis of Bell's arguments would be the best memorial in the honor of Walter Philipp.

1 Derivation of Bell's inequality

By the Kolmogorov axiomatics [23], see also [24], [25], [26] the *probability space* is a triple

$$\mathcal{P} = (\Omega, \mathcal{F}, \mathbf{P}),$$

where Ω is an arbitrary set, \mathcal{F} is an arbitrary σ -algebra¹ of subsets of Ω , \mathbf{P} is a σ -additive measure on \mathcal{F} which yields values in the segment $[0, 1]$ of the real line and normalized by the condition $\mathbf{P}(\Omega) = 1$.

Random variables on \mathcal{P} are by definition measurable functions

$$\xi : \Omega \rightarrow \mathbf{R}. \quad (1)$$

Thus $\xi^{-1}(B) \in \mathcal{F}$ for every $B \in \mathcal{B}$, where \mathcal{B} is the Borel σ -algebra on the real line.

Let $\mathcal{P} = (\Omega, \mathcal{F}, \mathbf{P})$ be a Kolmogorov probability space. For any pair of random variables $u(\omega), v(\omega)$, their covariation is defined by

$$\langle u, v \rangle = \text{cov}(u, v) = \int_{\Omega} u(\omega) v(\omega) d\mathbf{P}(\omega).$$

We reproduce the proof of Bell's inequality in the measure-theoretic framework.

Theorem 1. (Bell inequality for covariations) *Let $\xi_a, \xi_b, \xi_c = \pm 1$ be random variables on \mathcal{P} . Then Bell's inequality*

$$|\langle \xi_a, \xi_b \rangle - \langle \xi_c, \xi_b \rangle| \leq 1 - \langle \xi_a, \xi_c \rangle \quad (2)$$

holds.

Proof. Set $\Delta = \langle \xi_a, \xi_b \rangle - \langle \xi_c, \xi_b \rangle$. By linearity of Lebesgue integral we obtain

$$\Delta = \int_{\Omega} \xi_a(\omega) \xi_b(\omega) d\mathbf{P}(\omega) - \int_{\Omega} \xi_c(\omega) \xi_b(\omega) d\mathbf{P}(\omega) = \int_{\Omega} [\xi_a(\omega) - \xi_c(\omega)] \xi_b(\omega) d\mathbf{P}(\omega). \quad (3)$$

As

$$\xi_a(\omega)^2 = 1, \quad (4)$$

we have:

$$|\Delta| = \left| \int_{\Omega} [1 - \xi_a(\omega) \xi_c(\omega)] \xi_a(\omega) \xi_b(\omega) d\mathbf{P}(\omega) \right| \leq \int_{\Omega} [1 - \xi_a(\omega) \xi_c(\omega)] d\mathbf{P}(\omega). \quad (5)$$

¹In literature one also uses the terminology σ -field, instead of σ -algebra.

2 Correspondence between classical and quantum models

It is assumed that there exists a space of hidden variables Ω representing states of individual physical systems. There is fixed a σ -algebra \mathcal{F} of subsets of Ω . On this space there are defined *classical quantities*. These are measurable functions $\xi : \Omega \rightarrow \mathbf{R}$ – random variables on the measurable space (Ω, \mathcal{F}) .²

There is also considered a space of physical observables O . In the quantum model they are represented by self-adjoint operators, e.g., we can represent O by $\mathcal{L}_s(\mathcal{H})$ – the space of bounded self-adjoint operators in the Hilbert space of quantum states \mathcal{H} . We shall distinguish a physical observable and its operator-representative by using symbols: a and \hat{a} .

The main question is about existence of a correspondence between the space of random variables $V(\Omega)$ and the space of quantum observables $\mathcal{L}_s(\mathcal{H})$ – about the possibility to construct a map

$$j : V(\Omega) \rightarrow \mathcal{L}_s(\mathcal{H}) \quad (6)$$

or a map

$$i : \mathcal{L}_s(\mathcal{H}) \rightarrow V(\Omega) \quad (7)$$

which have “natural probabilistic properties” (in general j is not a one-to-one map; its existence does not imply existence of i and vice versa). The main problem is that physics does not tell us which features such maps should have. There is a huge place for mathematical fantasies (presented in the form of various NO-GO theorems). We now recall the history of this problem.

3 Von Neumann’s formalization

J. von Neumann was the first who presented a list of possible features of the classical→quantum map j , see [28]:

²In this framework classical is equivalent to existence of a *functional representation*. Denote the space of classical quantities by the symbol $V(\Omega)$. This is some space of real-valued (measurable) functions on Ω . The choice of this functional space depends on a model under consideration. For a system which state is given by the hidden variable ω , the value $\xi(\omega)$ of a classical quantity ξ gives the objective property ξ of this system. We shall not distinguish a classical (physical) quantity and its representation by random variable.

VN1). j is one-to-one map.³

VN2). For any Borel function $f : \mathbf{R} \rightarrow \mathbf{R}$, we have $j(f(\xi)) = f(j(\xi))$, $\xi \in V(\Omega)$.

VN3). $j(\xi_1 + \xi_2 + \dots) = j(\xi_1) + j(\xi_2) + \dots$ for any any sequence $\xi_k \in V(\Omega)$.⁴

Any statistical model contains a space of *statistical states*. In a pre-quantum statistical model (which we are looking for) statistical states are represented by probability measures on the space of hidden variables Ω . Denote such a space of probabilities by $S(\Omega)$. This space is chosen depending on a classical statistical model under consideration.⁵ In the quantum model statistical states are represented by von Neumann *density operators*. This space is denoted by $\mathcal{D}(\mathcal{H})$.

Roughly speaking J. von Neumann proved that under conditions VN1-VN3 every operation of statistical averaging on $V(\Omega)$ can be represented as the quantum trace-average on $\mathcal{L}_s(\mathcal{H})$ corresponding to a quantum state $\rho \in \mathcal{D}(\mathcal{H})$. By using the language of probability measures we can say that every probability measure \mathbf{P} on Ω can be represented by a quantum state ρ and vice versa. Thus we have the following “theorem” (von Neumann did not proceed in the rigorous mathematical framework):

Theorem 2. (von Neumann) *Under conditions VN1-VN3 (and some additional technical conditions) there is well defined map $j : S(\Omega) \rightarrow \mathcal{D}(\mathcal{H})$ which one-to-one and*

$$\int_{\Omega} \xi(\omega) d\mathbf{P}(\omega) = \text{Tr } \rho \hat{a}, \text{ where } \rho = j(\mathbf{P}), \hat{a} = j(\xi). \quad (8)$$

By using Theorem 2 J. von Neumann “proved” (he did not formulate a theorem, but just ansatz) [28]:

³Different random variables from the space $V(\Omega)$ are mapped into different quantum observables (injectivity) and any quantum observable corresponds to some random variable belonging $V(\Omega)$ (surjectivity).

⁴As J. von Neumann remarked: “the simultaneous measurability of $j(\xi_1), j(\xi_2), \dots$ is not assumed”, see [28], p. 314.

⁵For example, in classical statistical mechanics $S(\Omega)$ is the space of *all probability measures* on phase space $\Omega = \mathbf{R}^{2n}$. In a prequantum classical statistical field theory which was developed in a series of works [29],[30] the space of hidden variables Ω is infinite-dimensional phase space, space of classical fields, and the space of statistical states $S(\Omega)$ consists of Gaussian measures having very small dispersion.

Theorem 3. (Von Neumann) *Let the space of statistical states $S(\Omega)$ contains probabilities having zero dispersion. A correspondence map j between a classical statistical model*

$$M_{\text{cl}} = (S(\Omega), V(\Omega))$$

and the quantum statistical model

$$N_{\text{quant}} = (\mathcal{D}(\mathcal{H}), \mathcal{L}_s(\mathcal{H}))$$

satisfying the postulates V1–V3 (and some additional technical conditions [28]) does not exist.

4 Bell's NO-GO theorem

As was pointed out by many outstanding physicists (e.g., by J. Bell [1] and L. Ballentine [31]), some of von Neumann postulates of classical \rightarrow quantum correspondence are nonphysical. In the opposition to von Neumann, in Bell-type NO-GO theorem different classical quantities can correspond to the same quantum observable. It is not assumed that every self-adjoint operator corresponds to some classical quantity. It might be that some self-adjoint operators have no classical counterpart (or even physical meaning). The postulate V1 was deleted from the list for classical \rightarrow quantum correspondence. The most doubtful postulate V3 was also excluded from considerations. It was not assumed that the V2 holds.

We consider a family of spin operators:

$$\hat{\sigma}(\theta) = \cos \theta \hat{\sigma}_z + \sin \theta \hat{\sigma}_x,$$

where $\hat{\sigma}_x, \hat{\sigma}_z$ are Pauli matrices, $\theta \in [0, 2\pi)$. These operators act in the two dimensional state space $\mathcal{H} = \mathbf{C}^2$. We also consider spin operators for pairs of 1/2-spin particles: $\hat{\sigma}(\theta) \otimes I$ and $I \otimes \hat{\sigma}(\theta)$. They act in the four dimensional state space $\mathcal{H} = \mathbf{C}^2 \otimes \mathbf{C}^2$.

Bell's list of postulates on classical \rightarrow quantum correspondence can be described as following:

B1). The image $j(V(\Omega))$ contains spectral projectors for operators $\hat{\sigma}(\theta) \otimes I$ and $I \otimes \hat{\sigma}(\theta)$ for pairs of 1/2-spin particles.

B2). For any random variable $\xi \in V(\Omega)$, its range of values $\xi(\Omega)$ coincides with the spectrum of the operator $\hat{a} = j(\xi)$.

B3). The image $j(S(\Omega))$ contains the singlet spin state⁶

$$\psi = \frac{1}{\sqrt{2}}(|+ > |- > -|- > |+ >)$$

Starting with any classical→quantum mapping j we can construct a map i from quantum observables to classical random variables by setting for $\hat{a} \in j(V(\Omega))$,

$$i(\hat{a}) = \xi_a,$$

where ξ_a belongs to the set of random variables $j^{-1}(\hat{a})$. We also construct a map (denoted by the same symbol i) from the space of von Neumann density operators into the space of classical probability measures by choosing a probability measure \mathbf{P}_ρ belonging the set $j^{-1}(\rho)$. We emphasize that such maps

$$i : \mathcal{L}_s(\mathcal{H}) \rightarrow V(\Omega)$$

and

$$i : \mathcal{D}(\mathcal{H}) \rightarrow S(\Omega)$$

are not uniquely defined! We also point out to a purely mathematical problem of transition from classical→quantum to quantum→classical correspondence. Such a possibility is based on the *Axiom of Choice*. For example, for quantum observables we have the collection of sets $j^{-1}(\hat{a})$ of classical random variables corresponding to self-adjoint operators. We should choose from each of these sets one random variable and construct a new set – the classical image of quantum observables. The using of this axiom is not commonly accepted in the mathematical community.

We set

$$\xi_\theta = i(\hat{\sigma}(\theta) \otimes I), \quad \xi'_\theta = i(I \otimes \hat{\sigma}(\theta)).$$

These are classical pre-images of the spin operators for pairs of 1/2-spin particles.

J. Bell also proposed to use the following postulates:

B4). For any quantum state ρ and commuting operators \hat{a}, \hat{b} , the quantum and classical correlations coincide:

$$\langle \xi_a, \xi_b \rangle_{\mathbf{P}_\rho} \equiv \int_{\Omega} \xi_a(\omega) \xi_b(\omega) d\mathbf{P}_\rho(\omega) = \langle \hat{a} \hat{b} \rangle_\rho \equiv \text{Tr } \rho \hat{a} \hat{b}.$$

⁶This state belongs to the four dimensional state space $\mathcal{H} = \mathbf{C}^2 \otimes \mathbf{C}^2$. By using the tensor notations we can write $\psi = \frac{1}{\sqrt{2}}[|+ > \otimes |- > -|- > \otimes |+ >]$.

B5). For the singlet state ψ and any θ , random variables ξ_θ and ξ'_θ are anti-correlated:

$$\xi_\theta(\omega) = -\xi'_\theta(\omega) \quad (9)$$

almost everywhere with respect to the probability \mathbf{P}_ψ .

Theorem 4. (Bell) *Let $\dim \mathcal{H} = 4$. Correspondence maps*

$$j : V(\Omega) \rightarrow \mathcal{L}_s(\mathcal{H}) \text{ and } j : S(\Omega) \rightarrow \mathcal{D}(\mathcal{H}) \quad (10)$$

satisfying the postulates B1-B5 do not exist.

Proof. We apply Bell's inequality, Theorem 1, to random variables $\xi_\theta = i(\hat{\sigma}(\theta) \otimes I)$, $\xi'_\theta = i(I \otimes \hat{\sigma}(\theta))$ and to a probability measure \mathbf{P}_ψ corresponding to the singlet state ψ :

$$| \langle \xi_{\theta_1}, \xi_{\theta_2} \rangle_{\mathbf{P}_\psi} - \langle \xi_{\theta_3}, \xi_{\theta_2} \rangle_{\mathbf{P}_\psi} | \leq 1 - \langle \xi_{\theta_1}, \xi_{\theta_3} \rangle_{\mathbf{P}_\psi} .$$

We remark that the postulate B2 was used here. To prove Bell's inequality, we took into account that random variables $\xi_\theta(\omega) = \pm 1$. We now apply the anti-correlation postulate B5 and rewrite the Bell's inequality:

$$| \langle \xi_{\theta_1}, \xi'_{\theta_2} \rangle_{\mathbf{P}_\psi} - \langle \xi_{\theta_3}, \xi'_{\theta_2} \rangle_{\mathbf{P}_\psi} | \leq 1 + \langle \xi_{\theta_1}, \xi'_{\theta_3} \rangle_{\mathbf{P}_\psi} .$$

Finally, we apply the postulate B4 and write quantum covariations, instead of classical:

$$\begin{aligned} & | \text{Tr} (\psi \otimes \psi) (\hat{\sigma}(\theta_1) \otimes I) (I \otimes \hat{\sigma}(\theta_2)) - \text{Tr} (\psi \otimes \psi) (\hat{\sigma}(\theta_3) \otimes I) (I \otimes \hat{\sigma}(\theta_2)) | \\ & \leq 1 + \text{Tr} (\psi \otimes \psi) (\hat{\sigma}(\theta_1) \otimes I) (I \otimes \hat{\sigma}(\theta_3)) . \end{aligned}$$

It is well known that this inequality is violated for a special choice of angles $\theta_1, \theta_2, \theta_3$.

Our attitude with respect to Bell-type NO-GO theorems is similar to Bell's attitude with respect to others NO-GO theorems – von Neumann, Jauch-Piron and Gleasons theorems, – [1], p.4-9. As well as J. Bell did, we could speculate that some postulates about the correspondence between classical and quantum models (which were used in Bell-type NO-GO theorems) were nonphysical. There are many things which can be questioned in Bell's arguments.

5 The range of values postulate

The proofs of Bell and Wigner NO-GO theorems were based on the postulate B2 on the *coincidence of ranges of values* for classical random variables and quantum observables. Moreover, one can easily construct examples of classical random variables reproducing the EPR-Bohm correlations in the case of violation of B2, [32].

Is the postulate B2 really implied by the physical analysis of the situation? It seems that not at all! Henry Stapp pointed out [9]: ‘The problem, basically, is that to apply quantum theory, one must divide the fundamentally undefined physical world into two idealized parts, the observed and observing system, but *the theory gives no adequate description of connection between these two parts*. The probability function is a function of degrees of freedom of the microscopic observed system, whereas the probabilities it defines are probabilities of responses of macroscopic measuring devices, and these responses are described in terms of quite different degrees of freedom.’ In such a situation rejection of the range of values condition is quite natural, since, as was pointed by Stapp, a classical random variable ξ and its quantum counterpart $\hat{a} = j(\xi)$ depend on completely different degrees of freedom. Finally, we remark that a classical model reproducing quantum probabilistic description, but violating B2, was recently developed, see [29], [30].

Conclusion. *If the range of values postulates (in the forms V2, B2) are rejected, then the classical probabilistic description does not contradict quantum mechanics.*

6 Contextual viewpoint

In this section we shall present a very general viewpoint on the role of contextuality in Bell-type NO-GO theorems. Bell’s original viewpoint on contextuality was presented in [1]. The latter contextuality we can call *simultaneous measurement contextuality* or Bell-contextuality. We reserve the term *contextuality* for our general contextuality – dependence on the whole complex of physical conditions for preparation and measurement. We are aware that commonly in literature Bell-contextuality is called simply contextuality. However, using such a terminology is rather misleading, because dependence on the measurements of other compatible observables is just a very special case of dependence on the general physical context.

As was rightly pointed out by J. Bell, the only reasonable explanation of his contextuality is *action at the distance*. Another possibility is often called “*death of reality*” [6] – denying the possibility to assign to quantum systems objective properties (such as the electron spin or the photon polarization) – does not sound natural. The observation of precise (anti-)correlations for the singlet state evidently contradicts to the latter explanation.

In contrast to Bell-contextuality, in general contextuality does not imply action at the distance nor “death of reality.”

6.1 Non-injectivity of correspondence

We first concentrate our considerations on the *classical variables* \rightarrow *quantum observables* correspondence. As we remember, J. Bell (as well as L. Ballentine) criticized strongly the von Neumann postulate V1. Both Bell and Ballentine (as well as many others) emphasized that there were no physical reasons to suppose (as von Neumann did) that for a quantum observable \hat{a} its classical pre-image

$$j^{-1}(\hat{a}) = \{\xi \in V(\Omega) : j(\xi) = \hat{a}\}$$

should contain just one random variable.⁷

We now consider the *classical probabilities* \rightarrow *quantum states* correspondence. In the same way as for variables and observables there are no physical reasons to assume injectivity of the map $j : S(\Omega) \rightarrow \mathcal{D}(\mathcal{H})$. By saying that we prepared an ensemble of systems with the fixed quantum state ρ we could not guarantee that we really prepared the fixed classical probability distribution. The set

$$j^{-1}(\rho) = \{\mathbf{P} \in S(\Omega) : j(\mathbf{P}) = \rho\}$$

might have huge cardinality.

We remark that the derivations of all Bell-type NO-GO theorems were based on the possibility to select for any quantum state (at least for the singlet state) one fixed classical probability measure $\mathbf{P}_\rho \in j^{-1}(\rho)$ and for any quantum observable (at least for spin observables) the fixed random variable $\xi \in j^{-1}(\hat{a})$.

⁷ If one consider the quantum mechanical description as an *approximative description*, cf. [], then it would be quite reasonable to assume that quantum mechanics cannot distinguish sharply prequantum physical variables. A few different classical random variables ξ, η, \dots can be identified in the quantum model with the same operator $\hat{a} = j(\xi) = j(\eta) = \dots$

6.2 Contextual opposition against Bell's approach to NO-GO theorems

The crucial counterargument is that at the experimental level in all Bell-type inequalities one should use data which is obtained in a few different runs of measurements (at least three, but in the real experimental framework four), see [5], [27], [16]. In the light of the above discussion of the non-injectivity of classical \rightarrow quantum correspondence there are no physical reasons to assume that we would be able to obtain the same classical probability distribution and the same classical random variables (for example, corresponding to spin observables).⁸ We are not able to guarantee that all runs of measurements are performed under the same physical conditions.

Let us consider a new random variable C describing a *complex of physical conditions* (context) during a run of measurements. And let us try to proceed as J. Bell and his followers did by proving inequalities for correlations and probabilities. Now classical probability measures corresponding to a quantum state ρ (in particular, to the singlet state) depend on runs $C : \mathbf{P}_\rho \equiv \mathbf{P}_{\rho,C}$ as well as random variables: $\xi_{a,C}(\omega), \xi_{b,C}(\omega), \xi_{c,C}(\omega)$. We start with the correlation inequality. There are three different complexes of physical conditions C_1, C_2, C_3 inducing correlations which were considered in Theorem 1. Here

$$\begin{aligned} \langle \xi_a, \xi_b \rangle (C_1) &= \int_{\Omega} \xi_{a,C_1}(\omega) \xi_{b,C_1}(\omega) \mathbf{P}_{\rho,C_1}(\omega), \\ \langle \xi_c, \xi_b \rangle (C_2) &= \int_{\Omega} \xi_{c,C_2}(\omega) \xi_{b,C_2}(\omega) \mathbf{P}_{\rho,C_2}(\omega). \end{aligned}$$

If $C_1 \neq C_2$ we are not able to perform operations with integrals which we did in Theorem 1. We can not obtain the Bell's inequality involving the third correlation:

$$\langle \xi_a, \xi_c \rangle (C_3) = \int_{\Omega} \xi_{a,C_3}(\omega) \xi_{c,C_3}(\omega) \mathbf{P}_{\rho,C_3}(\omega)$$

for a context C_3 . To derive Bell's inequality, we should assume that $C_1 = C_2 = C_3$.

By using the contextual framework we derived in [34], [16] generalizations of the Bell-type inequalities. Such generalized inequalities do not contradict

⁸Even if we use the same macroscopic preparation and measurement devices, fluctuations of micro parameters can induce different physical conditions, [27], [16].

to predictions of quantum mechanics. We also mention that a special form of contextuality (so called *non-reproducibility condition*) was also present in arguments of De Baere [27] against Bell's NO-GO theorem, see also [16]. So called efficiency of detectors (or more general unfair sampling) argument [33] can also be considered as special forms of contextuality – different contexts produce samples with different statistical properties.

Conclusion. *The main value of Bell's arguments was the great stimulation of experimental technologies for working with entangled photons.*

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